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Josephson Lattices of the Optimal Size

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Abstract

The stability of the bound states of the magnetic flux in a Josephson resistive lattices is investigated numerically. It is shown that for a simple relationship between the geometrical parameters of the lattice the range of bias current is the widest.

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I. Formulation of the problem

In the present paper we study numerically stationary spatially periodic states of the magnetic flux in one-dimensional Josephson junctions, whose dielectric layer represents a lattice of resistive inhomogeneities of finite size.[1,2]

The mathematical model of such junctions is described by the perturbed stationary sin-Gordon equation

$$- \phi_{xx} + j_D(x) \sin \phi + \gamma = 0, \quad x \in (-R, R), \tag{1}$$

supplemented with the periodic boundary conditions

$$\phi(-R) = \phi(R), \quad \phi_x(-R) = \phi_x(R). \tag{2}$$

Here $j_D(x)$ is given continuous periodic function which models the distribution of the amplitude of the Josephson current along the junction; $2R$ is a given length of a spatial wave; γ is the bias current.

Let us consider in more details the choice of the function $j_D(x)$. For the homogeneous junctions we have $j_D(x) \equiv 1$. In an inhomogeneous case the function $j_D(x)$ is defined by the number of the inhomogeneities and their shape. The δ-function model of inhomogeneities is the most frequently used.[3,4,5,6,7,8,9] In this paper we use inhomogeneities of the shape of equilateral trapezium (see fig. 1). Such trapezium is characterized by the width of its bottom μ and its top $\sigma = m\mu$. In all cases the height of the trapezium equals to a dimensionless unit.[3]

For $m = 1$ the inhomogeneity is a rectangular with base μ. The more realistic model of an inhomogeneity is one with $m > 1$. The decay of the amplitude of the Josephson current from 0 to 1 in such an inhomogeneity takes place in the interval of length $\delta = \dfrac{m-1}{2}\, \mu$ for the particular choice of the width μ of the inhomogeneity.

Most of the calculations in this work were carried out for $m = 1.5$, i.e. for $\delta = 1/4$. We consider an "infinite" lattice consisting of the described inhomogeneities with separation Δ between their center positions. The simple relationship $\mu \leq \frac{\Delta}{m}$ between the values of Δ, μ and m ensures that two adjacent inhomogeneities do not overlap.

The lattice is symmetric when $\Delta - \sigma = \mu$; this yields $\mu = \dfrac{\Delta}{1+m}$.

To examine the stability of a certain solution $\phi(x)$ of nonlinear boundary value problem (BVP) (1), (2) we solve the Sturm-Liouville problem (SLP)

$$-\psi_{xx} + q(x)\psi = \lambda\psi, \tag{3}$$

$$\psi_x(\pm R) = 0, \tag{4}$$

$$\int_{-R}^{R} \psi^2(x)\, dx = 1, \tag{5}$$

where the potential

$$q(x) = j_D(x) \cos\phi(x) \tag{6}$$

is generated by this particular solution. The solution $\phi(x)$ is stable if the minimal eigenvalue $\lambda_{\min}$ of the SLP with the corresponding potential $q(x)$ given by (6), is positive.[3]

To solve numerically the nonlinear problems (1), 2 and 3 – (5) we use continuous analog of the Newton's method[10] with an optimal step.[11] At each iteration we discretize the corresponding linearized BVP using the spline-collocation second order scheme over a nonuniform grid with condensed spacing in the vicinity of the inhomogeneities. The detailed description of the algorithm and the numerical scheme is given in.[1]

II. Discussion of the numerical results

Each solution $\phi(x, \Delta, \mu, N_I, \gamma)$ of BVP (1), (2) depends on the geometrical parameters Δ, μ and N_I, and on the physical parameter γ as well. Here N_I is the number of inhomogeneities, enveloped by a single wave of length $2R = N_I\Delta$. Such waves exist in a real finite size junctions having N_I inhomogeneities symmetric with respect to its center. Here $2R$ is the length of the junction. We note that in this case the eq. (1) is supplemented by the boundary conditions of Neuman type

$$\phi_x(\pm R) = h_B. \tag{7}$$

Here h_B is the magnetic field at the boundaries of the junction.

The potential $q(x)$ of the SLP (3) - (5) is explicitly related to the periodic solution of the BVP (1), (2) by eq. (6), and, consequently, the corresponding eigenvalues and eigenfunctions depend on the parameters Δ, μ and N_I. Thus

the stability of the bound states of the magnetic flux in the junction also depends on these parameters.

Note that when the bias current $\gamma = 0$ the eqs. (1), (2) have "vacuum" (Meissner's) solutions of the form

$$\phi(x) = k\pi, \quad k = 0, \pm 1, \pm 2, \dots,$$

for every N_I. The solutions corresponding to even k are stable while the solutions corresponding to odd k are unstable.

Let us consider the influence of the geometric parameters on the possible bound states of the magnetic flux in the lattice.

For a junction with $N_I = 1$, $2R = \Delta < 2\pi$ and $\gamma = 0$ we find just two periodic solutions — stable and unstable Meissner's solutions. Such junction is "short" for periodic solutions, however, we should note that there are nontrivial stable aperiodic solutions corresponding to the eq. (1). The derivative $\phi_x(x)$ of one of these solutions is marked by "$\diamond$" in fig. 2. For large enough values of the distance Δ, nontrivial unstable periodic solutions exist in the junction. The most interesting among them is the one, whose derivative is marked by "Δ" in fig. 2. We will conditionally call this solution soliton. It remains unstable in a very wide range of the values Δ and μ; this is demonstrated in fig. 3 and fig. 4. The second unstable solution, marked by "$\Box$" in fig. 2, exists in a real junction with one inhomogeneity only for very large values of h_B at the boundaries.

There is a periodic stable solution in the lattice in the case $N_I = 2$. Further we refer to this solution as "main periodic soliton". This solution is marked by "o" in fig. 2 and fig. 5. If the two symmetric inhomogeneities of a junction are separated by large enough distance δ, then the periodic magnetic flux $\phi(x)$ is a result of the nonlinear interaction between two aperiodic stable pinned at the inhomogeneities structures - the fluxon and the antifluxon. Similarly, for the case $N_I = 1$ the unstable solutions are the result of "interactions" between two unstable fluxons, alienated from the inhomogeneity. For these reasons one can make the conjecture that for the case $N_I = 1$ there aren't any nontrivial periodic stable solutions.[12]

The solution, marked by "∇" in fig. 5 (the "secondary" periodic soliton) is an example of weakly stable bound state, whose energy is approximately four times greater then the energy of the main periodic soliton. The stable soliton with small $\lambda_{\min}$ exists only for very large values of the parameters Δ and μ of the lattice. Both solitons have symmetric reflections about the

axis x. Further we refer to them as main periodic antisoliton and secondary periodic antisoliton. The remaining two solutions in fig. 5, marked by "$\Box$" and "$\diamond$" are unstable.

Fig. 6 shows the minimal eigenvalue $\lambda_{\min}$, corresponding to two stable periodic solutions, as a function of the parameter μ when $N_I = 2$ and bias current $\gamma = 0$. Different curves correspond to different values of the distance Δ between the inhomogeneities. It is very important to note, that all the curves have maximum in a certain point μ_c, dependent on the parameter Δ. This leads that in a lattice with inhomogeneities separated by the distance Δ with width μ_c, the time-dependent perturbations of the magnetic flux will decay smoothly.[1] The location of the points of maximum μ_c for all the solutions depends on the value of Δ and for the main soliton (as for main antisoliton) is equal approximately to 1.3.

Apart from the periodic stable solutions in a real junction with two inhomogeneities there are stable aperiodic bound states of the magnetic flux when the boundary magnetic field $h_B = 0$; this case corresponds to the boundary conditions (7).

The distribution of the aperiodic magnetic field $\phi_x(x)$ along the junction is shown in fig. 7 (the symmetric reflections are not shown). Here the values of the parameters are $R = 6$, $N_I = 2$, $h_B = 0$, $\gamma = 0$. The marked by "o" solution is periodic, others are not. The energy of the "complex" bound state (periodic soliton), pinned at two inhomogeneities, is much greater than the energy of the aperiodic solitons. The minimal eigenvalue of the periodic soliton is smaller than the minimal eigenvalues corresponding to the aperiodic bound states.

The minimal eigenvalue $\lambda_{\min}$ as a function of the parameter μ has an maximum for the aperiodic bound states as well as for the aperiodic ones. This is demonstrated in fig. 8.

For comparison, the same curve (marked by "$\diamond$"), corresponding to the stable fluxon in the junction having one inhomogeneity, is shown in fig. 9.

The curves $\lambda_{\min}(\mu)$, corresponding to the unstable periodic solutions, are shown in fig. 9. Here the parameters of the lattice are N_I, $\mu = 2.4$ and the bias current $\gamma = 0$. The stable main periodic soliton transforms into a unstable solution at the point of bifurcation B_0 (the unstable soliton is marked by "$\Box$" in fig. 5). The secondary periodic soliton is unstable in the lattice with so "narrow" inhomogeneities. The secondary soliton transforms into another unstable periodic soliton at the point of bifurcation B_1 when

the second eigenvalue of the eqs. (3) - (5) is also equal to zero. The latter unstable soliton is marked by "$\diamond$" in fig. 5. Increasing of the width μ of the inhomogeneities "lifts" the curve $\lambda_{\min}(\mu)$ upwards and, if μ is greater than a certain critical value, the secondary periodic soliton becomes stable.

Let us consider the dependence of the bound states of the magnetic flux on the bias current γ. It is well-known that in the inhomogeneous junction the stable solutions lose their stability when the absolute value of the bias current is increased. In the periodic case the process of "destruction" of the magnetic field by the bias current is demonstrated in fig. 10. It can be seen,[13] that the positive values of the current γ "compress" the field in the center between two inhomogeneities, while the negative values of γ push it to the adjacent inhomogeneities or, in the case of a real junction with two inhomogeneities, to the boundaries of the junction.

The curves $\lambda_{\min}(\gamma)$ for all found stable solutions as well as unstable, in the case $N_I = 2$, $\Delta = 6$, $\mu = 2.4$ are demonstrated in fig. 11. The curves, corresponding to the stable solutions, have their symmetric reflections about the line $\gamma = 0$. As before, B_0 denotes the bifurcation points, corresponding to the roots of the equation $\lambda_{\min}(\gamma) = 0$.

Let γ_a and γ_b are the roots of this equation (the points B_{0a} and B_{0b} correspond to this roots in fig. 11 for the main soliton/antisoliton). Let us designate the difference

$$\Delta\gamma = \gamma_b - \gamma_a$$

interval of stability of the main periodic soliton with respect to the bias current γ. Generally, the value of $\Delta\gamma$ is a function of the parameters Δ and μ. The interval of stability $\Delta\gamma$ for the main soliton as a function of the parameter μ, when the value of the distance Δ is fixed, is shown in the fig. 12. These curves can be approximated by the cubic polynomials with coefficients, depending only on Δ, in the whole domain where μ is defined physically. For a given Δ the points of maximum μ_s determine the width of the inhomogeneities, so, that the soliton has the maximal interval of stability $\Delta\gamma$ with respect to the current γ.

Note, that the shape of the curve $\mu_s(\Delta)$ can be fitted by a line $\mu_s(\Delta) = a\Delta + b$ with a good accuracy. This is the line, marked by "o" in fig. 13. The approximate values of the parameters of this line, calculated numerically, are: $a \approx 0.254$, $b \approx -8.5.10^{-3}$.

Having in mind the above results we can construct an "optimal" junction

with geometric parameters related by

$$\mu_c = \mu_s(\Delta_{opt}).$$

In this case we have $N_I = 2$, $\Delta_{opt} \approx 4\mu_c$. Thus, the "optimal" distance between the inhomogeneities is $\Delta_{opt} \approx 5.2$.

III. Conclusions

We show numerically that in the Josephson junctions with resistive inhomogeneities the stable periodic bound states of the magnetic flux, pinned on the inhomogeneities, have a certain minimal wavelength. The dependence of the stable bound states on the width μ of the inhomogeneities and the distance Δ between them is studied. In particular, it is demonstrated that the curve $\lambda_{\min}(\mu)$ for a fixed Δ has maximum μ_c, i.e. in such a junction the time-dependent perturbations of the magnetic flux decays smoothly. The dependence of the interval of stability $\Delta\gamma$ on the parameter μ is also studied. For a certain relationship between the with of the inhomogeneities and their separation ("optimal" lattices) the interval of stability $\Delta\gamma$ is the widest.

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V. Figure captions

1. Geometrical model of a resistive inhomogeneity.

2. Some distributions of the magnetic field $\phi_x(x)$ along the junction. The solution, marked by "$\diamond$", is stable aperiodic soliton of the magnetic field in the case $N_I = 1$; the solutions, marked by "$\square$" and "Δ" are unstable periodic (case $N_I = 1$); marked by "o" solution is stable periodic (case $N_I = 2$).

3. Minimal Eigenvalues $\lambda_{\min}$ of the SLP (3) - (5) versus the distance Δ between the inhomogeneities for the solutions, marked by "$\square$" and "Δ" in fig. 2.

4. Minimal Eigenvalues $\lambda_{\min}$ of the SLP (3) - (5) versus the width μ of the inhomogeneities for the solutions, marked by "$\square$" and "Δ" in fig. 2.

5. Stable (marked by "$\diamond$" and "$\square$") and unstable (marked by "$\diamond$" and "$\square$") distributions of the magnetic field $\phi_x(x)$ along the junction with parameters $N_I = 2$, $\Delta = 9.5$, $\mu = 2.4$. Here $\gamma = 0$.

6. Minimal eigenvalue $\lambda_{\min}$ of the SLP (3) - (5) versus the width μ of the inhomogeneities for the stable solutions from fig. 5. Here the parameter Δ is fixed and the point μ_c corresponds to the "optimal" width.

7. Stable bound states of the magnetic field $\phi_x(x)$ corresponding to the aperiodic boundary conditions (7). Here $N_I = 2$, $\Delta = 6$, $\mu = 1$, $\gamma = 0$, $h_B = 0$. The quantity $\Delta\phi = \phi(R) - \phi(-R)$ is the full magnetic flux through the junction.

8. Minimal eigenvalue $\lambda_{\min}$ of the SLP (3) - (5) versus the width μ of the inhomogeneities for the stable solutions from fig. 5. Here the parameter Δ is fixed.

9. Minimal eigenvalue $\lambda_{\min}$ of the SLP (3) - (5) versus the distance Δ between the inhomogeneities. B_0 and B_1 are the points of bifurcation.

10. Variation of the shape of the stable soliton as the bias current is increased. Here $N_I = 2$, $\Delta = 6$, $\mu = 2.4$.

11. Minimal eigenvalue $\lambda_{\min}$ of the SLP (3) - (5) versus the bias current γ. B_{0a} and B_{0b} — for the periodic soliton from fig. 7.

12. Interval of stability $\Delta\gamma = \gamma_b - \gamma_a$ versus the width μ of the inhomogeneities. The parameter Δ is fixed.

13. The width μ_s, corresponding to the largest interval of stability versus the distance Δ in the cases $N_I = 2$ and $N_I = 3$.

References

[1]N.V. Alexeeva, T.L. Boyadjiev, Bulgarian Journal of Physics, v. 24, 1997, No 1–2, p. 63.

[2]T.L. Boyadjiev, N.V. Alexeeva, Comm. JINR, P17–97–28, Dubna, 1997 (In Russian).

[3]Yu. S. Gal'pern, A.T. Filippov, JETF (Soviet Journ. Teor. Exper. Fiz), v. 86, No 4, 1984, p. 1527 (In Russian).

[4]I.L. Serpuchenko, A.V. Ustinov, Solid State Comm., 68, 7, 1988, p. 693.

[5]B.A. Malomed, A.V. Ustinov, Fizika Nizkih Temp., 15, (1989), p. 1128.

[6]B.A. Malomed, A.V. Ustinov, Phys. Rev. B, v. 49, No 18 (1994), p. 13024.

[7]R. Fehrenbacher, V.B. Geshkenbein, G. Blatter, Phys. Rev. B, v. 45, No 10 (1992), p. 5450.

[8]B.H. Larsen, J. Mygind, A.V. Ustinov, Phys. Lett. A, 193 (1994), p. 359.

[9]A.V. Ustinov, Josephson vortices in distributed superconducting structures, D.Sc. Thesis, Tchernogolovka, 1994 (In Russian).

[10]E.P. Jidkov, G.I. Makarenko, I.V. Puzynin, In "Physics of elementary particles and atomic nucleus", Moscow, Atomizdat, 1973, v. 4, No 1, p. 127 (In Russian).

[11]V.V. Ermakov, N. N. Kalitkin, JVMiMF, v. 21, No 2, 1981 (In Russian).

[12]A.V. Filippov, 1992 (private communication) .

[13]D.W. McLaughlin, A.C. Scott, Phys. Rev. A, 18 (1978), p. 1652.